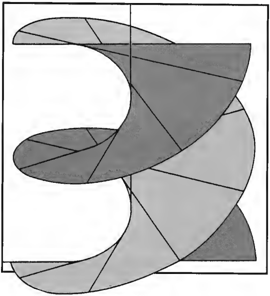
M434 Part VI
Mathematics: A Fourth Level Course

M434
DIFFERENTIAL GEOMETRY



PART VI
GEOMETRY OF SURFACES

The Open University

Mathematics: A Fourth Level Course

M434 Differential Geometry

Part VI Geometry of Surfaces

Prepared for the Course Team
by Bob Margolis

Set book

Barrett O'Neill, *Elementary Differential Geometry*, hardback edition (Academic Press, 1966).

It is essential to have this book; the course is based on it and will not make sense without it.

The set book is referred to as *O'Neill*.

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Introduction

Our study of surfaces has not made direct use of differential forms, nor of the connection and structural equations which we took so much trouble to develop earlier. We have pointed out the links with the equations from time to time, now we summarize these links and pursue the 'form' approach a little further.

There are two main themes. Firstly, we shall show how differential forms on a surface can be used to calculate Gaussian curvature and, hence, how they may be used to *define* Gaussian curvature for surfaces in general. Secondly, we shall return to the Gauss map (from a surface to the unit sphere) and link it to area 2-forms and Gaussian curvature.

The first two sections are based on reading *O'Neill*, Sections 1 and 2. They form the beginnings of *O'Neill*'s program of investigating surfaces from the point of view of inhabitants of the surface who know nothing about $\mathbf{E}^3$.

Section 3 is based on Section 6 of *O'Neill*. This section does not depend, in any essential way, on the three omitted sections of *O'Neill*. It contains useful, practical methods for calculating Gaussian curvature.

Section 4 does not have a reading passage from *O'Neill*. It pulls together the Gauss map, area 2-forms and the 'pullback' of the Gauss map.

The pullback of the mapping F is F^*, as defined in *Part IV*, Section 5.

Section 5 is a brief summary.

Study advice

The final TMA for the course has questions on Sections 4–6 of *Part V* and all of *Part VI*. The following represents a possible plan for study weeks.

Week 1 *O'Neill*, Chapter V, Sections 4 and 5.

Week 2 *O'Neill*, Chapter V, Section 6 and Chapter VI, Section 1.

Week 3 *O'Neill*, Chapter VI, Sections 2 and 3.

Week 4 Sections 4 and 5 of this text and TMA04.

1 The fundamental equations

Read *O'Neill, Chapter VI, Section 1, pages 245–250.*

This section draws together a number of the remarks that we have made about the links between the shape operator, defined in terms of a unit normal vector field, and the connection forms of *Part II*.

Throughout this commentary, we shall assume that E_1, E_2 and E_3 is an adapted frame field on at least part of a surface M.

Dual 1-forms The dual 1-forms for a general frame field are completely determined by the equations

$$\theta_i(E_j) = \delta_{ij}, \quad i, j = 1, 2, 3.$$

The remark on page 248, that $\theta_3 = 0$, is a little peculiar, for a tangent vector may be attached to a surface in $\mathbf{E}^3$ without, necessarily, being tangent to the surface. Such a vector may well have a non-zero E_3-component.

However, the purpose of this section is to begin disposing of all the concepts that belong to the surface because it lies in $\mathbf{E}^3$. Tangents to the surface will be retained, normals, including E_3 will not. Thus θ_3 disappears along with E_3!

Remark Once we have the dual 1-forms, all 1-forms on the surface can be expressed as linear combinations of θ_1 and θ_2, including the connection forms. We shall use this fact in Section 3.

Similarly, all 2-forms on the surface can be expressed in terms of the wedge product $\theta_1 \wedge \theta_2$.

Theorem 1.7 As with many such results in *O'Neill*, the proof consists of writing out the connection and structural equations in full, extracting the particular components required and then substituting any special information. Here the last step is to use $\theta_3 = 0$.

As an example and reminder, we have

$$d\begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} = \begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix}.$$

Matrix multiplication and substitution of

$$\theta_3 = d\theta_3 = 0$$

give precisely the first three results of the theorem.

Exercise 1.1 Suppose that E_1, E_2 and E_3 form an adapted frame field on the surface $M \subset \mathbf{E}^3$. Suppose also that α is a curve in M such that the velocity of α is the restriction of E_1 to α. That is,

$$E_1(\alpha(t)) = \alpha'(t).$$

(a) Explain why, with the usual notation, $\nabla_{E_1} V = V'$, for any vector field V on α.

(b) Show that α is a geodesic if, and only if, $\omega_{12}(E_1) = 0$. (*Hint*: Apply the definition of *geodesic* and the result of part (a).)

Exercise 1.2 This question uses the same notation as the previous exercise. Show that the normal curvature along α is $\omega_{13}(E_1)$.

Exercise 1.3 Let T be the torus parametrized by

$$\mathbf{x}(u, v) = ((R + r\cos u)\cos v, (R + r\cos u)\sin v, r\sin u), \quad R > r > 0.$$

This question indulges in the usual abuse of notation in regarding u and v as being defined on T rather than on the domain of $\mathbf{x}$.

(a) Show that the partial velocities can be used to define orthogonal unit tangent vector fields E_1, E_2 as follows:

$$E_1 = \frac{\mathbf{x}_u}{\|\mathbf{x}_u\|}, \quad E_2 = \frac{\mathbf{x}_v}{\|\mathbf{x}_v\|}.$$

Define the third member E_3 of an adapted frame field on T.

(b) Calculate

$$du(E_1), \quad du(E_2), \quad dv(E_1) \quad \text{and} \quad dv(E_2).$$

Hence show that the dual 1-forms are given by

$$\theta_1 = r\,du, \quad \theta_2 = (R + r\cos u)dv.$$

(c) By direct calculation show that

$$d\theta_1 = 0, \quad d\theta_2 = -r\sin u\, du\, dv$$

and deduce that

$$d\theta_2 = -\frac{\sin u}{R + r\cos u}\theta_1 \wedge \theta_2.$$

(d) Since ω_{12} is a 1-form on T, it is a linear combination

$$\omega_{12} = a\theta_1 + b\theta_2$$

of the dual 1-forms. By substituting into the first structural equations (Lemma 1.7(1)), show that

$$\omega_{12} = -\frac{\sin u}{R + r\cos u}\theta_2.$$

(e) Show that

$$d\omega_{12} = -\frac{\cos u}{r(R + r\cos u)}\theta_1 \wedge \theta_2.$$

(*Hint*: When differentiating the expression for ω_{12} you will need the Leibniz property and the value of $d\theta_2$ already found.)

[*Solutions on page 12*]

2 Form computations

Read *O'Neill, Chapter VI, Section 2, pages 251–255.*

Erratum *O'Neill*, page 252, the calculation of $S(E_1)$, $S(E_2)$ and the two lines following should read:

$$\begin{aligned} S(E_1) &= -\nabla_{E_1} E_1 \\ &= -(\omega_{31}(E_1)\, E_1 + \omega_{32}(E_1)\, E_2) \\ &= \omega_{13}(E_1)\, E_1 + \omega_{23}(E_1)\, E_2 \quad \text{(alternation)} \end{aligned}$$

$$\begin{aligned} S(E_2) &= -\nabla_{E_2} E_1 \\ &= -(\omega_{31}(E_2)\, E_1 + \omega_{32}(E_2)\, E_2) \\ &= \omega_{13}(E_2)\, E_1 + \omega_{23}(E_2)\, E_2 \quad \text{(alternation).} \end{aligned}$$

Thus the matrix of S with respect to $\{E_1, E_2\}$ is

$$\begin{pmatrix} \omega_{13}(E_1) & \omega_{13}(E_2) \\ \omega_{23}(E_1) & \omega_{23}(E_2) \end{pmatrix}.$$

■

This section gathers together a number of the ideas on which we have already remarked in the commentaries to earlier chapters of *O'Neill*.

In *Part V*, Section 3 we obtained the matrix on page 252 and remarked that

$$K = (\omega_{13} \wedge \omega_{23})(E_1, E_2) = -d\omega_{12}(E_1, E_2).$$

The observations in Lemma 2.1 immediately give Corollary 2.3.

It is important to realize that the significance of Corollary 2.3 is not so much that it is yet another formula for calculating Gaussian curvature but that it contains only forms that belong to the geometry of the surface. None of the items in the formula

$$d\omega_{12} = -K\theta_1 \wedge \theta_2$$

are dependent on the surface being in $\mathbf{E}^3$. It follows that, if we want to generalize our work to abstract surfaces, Corollary 2.3 can become the *definition* of Gaussian curvature.

This is not an unfamiliar process. We have already used a lemma:

$$\alpha'[f] = (f(\alpha))',$$

about directional derivatives of functions with respect to the velocity of a curve in E^3, to formulate the definition of directional derivative for functions defined on surfaces.

These exercises continue tbe investigation of the torus begun in the second exercise of the last section. You should use the results obtained there.

Exercise 2.1 Write down the Gaussian curvature of the torus T.

Exercise 2.2 Is the frame field (in Exercise 2.1) principal?

[*Solutions on page 13*]

3 Orthogonal coordinates

Read *O'Neill, Chapter VI, Section 6, pages 276–279.*

This section generalizes the work you have done (in various exercises) on the torus.

The feature that was responsible for simplifying many of the calculations on the torus was the orthogonality of $\mathbf{x}_u$ and $\mathbf{x}_v$. The fact that the parameter curves were principal was merely a bonus.

If you inspect the calculations in this section carefully and compare them with the solutions to the torus exercises, you should see the analogies.

Note how, at the top of page 277, *O'Neill* finally agrees that the abuse of notation that we have been committing is reasonable.

Finding ω_{12} If you cannot see how *O'Neill* did the 'comparison with the first structural equations' on page 277, we suggest the following systematic approach.

Since ω_{12} is a 1-form, it can be written

$$\omega_{12} = a\,du + b\,dv,$$

for some functions a and b on M.

Thus

$$\begin{aligned}
-\frac{(\sqrt{E})_u}{\sqrt{G}} du \wedge \theta_2 = d\theta_1 \\
&= \omega_{12} \wedge \theta_2 \\
&= (a\,du + b\,dv) \wedge \sqrt{G} dv \\
&= a\sqrt{G}\,du\,dv \\
&= a\,du \wedge \theta_2.
\end{aligned}$$

It follows that

$$a = -\frac{(\sqrt{E})_u}{\sqrt{G}}.$$

The value of b is obtained from

$$d\theta_2 = -\omega_{12} \wedge \theta_1$$

in exactly the same way.

Lemma 6.3 We think that *O'Neill* gives very sound advice. It is the derivation of this lemma that is important. For any particular example it is better to work directly from the structural equations than to memorize the result of Lemma 6.3.

When following this advice there is a choice to be made: whether to work in terms of du and dv or the dual 1-forms θ_1 and θ_2.

The calculation of the exterior derivatives is slightly easier in terms of du and dv because of the simple formulas

$$df = f_u\,du + f_v\,dv$$
$$d(df) = 0.$$

It is theoretically more 'elegant' to work in terms of the dual 1-forms.

The main difference comes when calculating $d\omega_{12}$. Suppose that

$$\omega_{12} = a\,du + b\,dv = f\theta_1 + g\theta_2.$$

Then

$$\begin{aligned} d\omega_{12} &= (da\,du + a\,d(du)) + (db\,dv + b\,d(dv)) \\ &= (a_u\,du + a_v\,dv)du + (b_u\,du + b_v\,dv)dv \\ &= a_v\,dv\,du + b_u\,du\,dv \\ &= (b_u - a_v)du\,dv, \end{aligned}$$

Using the other expansion gives

$$\begin{aligned} d\omega_{12} &= (df \wedge \theta_1 + f\,d\theta_1) + (dg \wedge \theta_2 + g\,d\theta_2) \\ &= df \wedge \theta_1 + dg \wedge \theta_2 + f\,d\theta_1 + g\,d\theta_2 \\ &= df \wedge \theta_1 + dg \wedge \theta_2 + f\,\omega_{12} \wedge \theta_2 - g\,\omega_{12} \wedge \theta_1 \\ &= df \wedge \theta_1 + dg \wedge \theta_2 + \omega_{12} \wedge (f\theta_2 - g\theta_1) \\ &= df \wedge \theta_1 + dg \wedge \theta_2 + (f\theta_1 + g\theta_2) \wedge (f\theta_2 - g\theta_1) \\ &= df \wedge \theta_1 + dg \wedge \theta_2 + (f^2 - g^2)\theta_1 \wedge \theta_2. \end{aligned}$$

There is a return for taking the extra trouble to work in terms of the dual 1-forms. Since df and dg are 1-forms, they can be expressed in terms of θ_1 and θ_2. Thus all terms in the expression for $d\omega_{12}$ automatically appear in terms of $\theta_1 \wedge \theta_2$ which enables you to read off the Gaussian curvature from

$$d\omega_{12} = -K\theta_1 \wedge \theta_2.$$

We continue the investigation of the torus with the usual parametrization.

Exercise 3.1 Calculate E, F and G for the standard parametrization of the torus and hence calculate K from the formula in Lemma 6.3.

Exercise 3.2 This exercise is an alternative approach to Exercise 3.1.

(a) Using the formulas on page 277, write down expressions for the dual 1-forms in terms of du and dv.

(b) Calculate $d\theta_1$ and $d\theta_2$ in terms of $du\,dv$.

(c) Assume that $\omega_{12} = a\,du + b\,dv$. and use the first structural equation to find a and b.

(d) Calculate $d\omega_{12}$ in terms of $du\,dv$.

(e) Use the second structural equation to calculate the Gaussian curvature K.

[Solutions on page 13]

4 The Gauss map

There is no reading from *O'Neill* for this section. We now take up the second theme of this part: the link between the Gauss map, area 2-forms and Gaussian curvature in the rather special case of a compact orientable surface.

Chapter VII of *O'Neill* does deal with more general considerations.

We assume the following situation. We have a surface M parametrized by a single mapping $\mathbf{x}(u,v)$ and the unit sphere Σ. We shall define a parametrization of the sphere shortly.

We assume that M has a unit normal vector field U obtained from

$$U(\mathbf{x}(u,v)) = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}.$$

The Gauss map F from M to Σ is defined by

$$F(\mathbf{x}(u,v)) = U(\mathbf{x}(u,v)),$$

where we interpret $U(\mathbf{x}(u,v))$ as giving the coordinates of a point on the unit sphere. Since $U(\mathbf{x}(u,v))$ is unit length, its coordinate functions do define a point on the unit sphere.

Now we define a parametrization of the sphere by

$$\mathbf{y}(u,v) = U(\mathbf{x}(u,v)) = F(\mathbf{x}(u,v)).$$

The sphere Σ also has a unit normal vector field V. Because it is a unit sphere, we can see that V has the same coordinate functions as $\mathbf{y}$:

$$V(\mathbf{y}(u,v)) = \mathbf{y}(u,v) = U(\mathbf{x}(u,v)) = F(\mathbf{x}(u,v)).$$

If we ignore points of application, $U = V$; more precisely, U and V are parallel for all values of u and v.

The Gauss map has a number of interesting properties. We consider two of them. Firstly, because, by definition

$$\mathbf{y}(u,v) = F(\mathbf{x}(u,v)),$$

we have

Here the special case of the effect of derived maps on page 161 of *O'Neill is* applicable.

$$\mathbf{y}_u(u,v) = F_*(\mathbf{x}_u(u,v)),$$
$$\mathbf{y}_v(u,v) = F_*(\mathbf{x}_v(u,v)).$$

Secondly, if we ignore points of application in a thoroughly cavalier fashion,

$$\begin{aligned} F_*(\mathbf{x}_u(u,v)) &= \frac{\partial}{\partial u}F(\mathbf{x}(u,v)) \\ &= \frac{\partial}{\partial u}U(\mathbf{x}(u,v)) \\ &= U_u(\mathbf{x}(u,v)) \\ &= -S(\mathbf{x}_u(u,v)). \end{aligned}$$

More precisely, we could say that $F_*(\mathbf{x}_u(u,v))$ and $-S(\mathbf{x}_u(u,v))$ are parallel for all u and v. Thus $\mathbf{y}_u(u,v)$ is parallel to $-S(\mathbf{x}_u(u,v))$ and similarly for the other partial velocity.

Next we define area 2-forms on both surfaces. We need only define their effect on the partial velocities as the partial velocities form a basis for the tangent plane at each point.

On M we define the 2-form η by

η pronounced 'eta'.

$$\eta(\mathbf{x}_u, \mathbf{x}_v) = \mathbf{x}_u \times \mathbf{x}_v \cdot U$$

and on Σ we define ζ by

ζ pronounced 'zeta'.

$$\zeta(\mathbf{y}_u, \mathbf{y}_v) = \mathbf{y}_u \times \mathbf{y}_v \cdot V.$$

We now have two 2-forms on M, the area 2-form η defined above and the pullback of ζ from Σ to M via the Gauss map F. This second 2-form is defined by

Part IV, Section 5.

$$(F^*\zeta)(\mathbf{x}_u, \mathbf{x}_v) = \zeta(F_*(\mathbf{x}_u), F_*(\mathbf{x}_v)).$$

We now explore the relation between $F^*\zeta$ and η.

We can calculate the effect of η and ζ on the appropriate partial velocities quite easily.

$$\begin{aligned}\eta(\mathbf{x}_u, \mathbf{x}_v) &= \mathbf{x}_u \times \mathbf{x}_v \cdot U \\ &= \|\mathbf{x}_u \times \mathbf{x}_v\| U \cdot U \quad \text{(definition of } U) \\ &= \|\mathbf{x}_u \times \mathbf{x}_v\|.\end{aligned}$$

Suppose that

$$\begin{aligned}S(\mathbf{x}_u) &= a\mathbf{x}_u + b\mathbf{x}_v, \\ S(\mathbf{x}_v) &= c\mathbf{x}_u + d\mathbf{x}_v,\end{aligned}$$

for suitable functions a, b, c, d.

Then

$$\begin{aligned}\zeta(\mathbf{y}_u, \mathbf{y}_v) &= \mathbf{y}_u \times \mathbf{y}_v \cdot V \\ &= (-S(\mathbf{x}_u)) \times (-S(\mathbf{x}_v)) \cdot U \\ &= (a\mathbf{x}_u + b\mathbf{x}_v) \times (c\mathbf{x}_u + d\mathbf{x}_v) \cdot U \\ &= (ad - bc)\mathbf{x}_u \times \mathbf{x}_v \cdot U \\ &= (ad - bc)\|\mathbf{x}_u \times \mathbf{x}_v\|.\end{aligned}$$

But $ad - bc$ is just the determinant of the shape operator S, that is the Gaussian curvature. Hence

$$\zeta(\mathbf{y}_u, \mathbf{y}_v) = K\eta(\mathbf{x}_u, \mathbf{x}_v).$$

Now consider the pullback $F^*\zeta$.

$$\begin{aligned}F^*\zeta(\mathbf{x}_u, \mathbf{x}_v) &= \zeta(F_*(\mathbf{x}_u), F_*(\mathbf{x}_v)) \\ &= \zeta(\mathbf{y}_u, \mathbf{y}_v) \\ &= K\eta(\mathbf{x}_u, \mathbf{x}_v).\end{aligned}$$

We have shown that $F^*\zeta$ and $K\eta$ agree on the linearly independent pair $(\mathbf{x}_u, \mathbf{x}_v)$ at each point so they are equal as 2-forms. That is

$$F^*\zeta = K\eta.$$

This provides yet another approach to finding Gaussian curvature. However, it is only in some simple cases that such an approach is less trouble than previous ones.

Perhaps it is of more interest because it generalizes the observation made earlier about what happens to areas under mappings. In the simple case of a matrix mapping from the plane to itself, it is the determinant of the matrix that controls how areas are mapped. In the mapping of a surface to a sphere, it is the determinant of the shape operator that controls mapping of area 2-forms.

Remark: You may have worried that there are two choices for the normal on the surface and also on the sphere, hence two area 2-forms on each surface. You may care to check that, provided we always define the Gauss map by using the chosen normal on M, and hence define $V = U$, the result is exactly the same.

We set one, simple, application of the above.

Exercise 4.1 Let T be the torus parametrized by

$$\mathbf{x}(u, v) = ((R + r\cos u)\cos v, (R + r\cos u)\sin v, r\sin u), \qquad R > r > 0$$

and Σ be the unit sphere parametrized by

$$\mathbf{y}(u, v) = (\cos u \cos v, \cos u \sin v, \sin u).$$

Let η and ζ be their respective area 2-forms.

(a) Use the partial velocities to define an outward unit normal vector field U on T.

(b) Show that, with the above choice of U,

$$F : \mathbf{x}(u, v) \longmapsto \mathbf{y}(u, v)$$

is the Gauss map of the torus to the sphere.

(c) Calculate directly

$$\eta(\mathbf{x}_u(u, v), \mathbf{x}_v(u, v))$$

and

$$\zeta(\mathbf{y}_u(u, v), \mathbf{y}_v(u, v)).$$

(d) Calculate

$$F^*\zeta(\mathbf{x}_u(u, v), \mathbf{x}_v(u, v)).$$

Hence deduce the Gaussian curvature of the torus.

[*Solution on page 14*]

5 Summary

This text has mainly been concerned with tying together some of the threads that have run through the course. We have shown how the shape operator ties in with the connection forms and indicated how this link might be used to define Gaussian curvature in more general work.

There is more in *O'Neill* than is appropriate to include in a half-credit course. We have tried to ensure that the material included is coherent, whilst still providing you with a sound basis for further reading.

Underlying all the theorems and computational techniques are the ideas from Chapter I, particularly the linearity and Leibniz properties of the various types of derivative.

You may well have found that the amount of new notation in the early chapters was somewhat daunting; by now the reasons for introducing it all should be clearer. Differential forms, in particular, provide a notationally cleaner approach than the use of tensors that you may find in earlier textbooks. The difference is similar to using matrices, rather than always working with coordinates, in linear algebra.

You should now be able to calculate the 'curvature apparatus' for curves and surfaces with some confidence. Although there appear to be quite a large number of techniques, they are all based upon use of a small collection of principles.

Solutions to the exercises

Solution 1.1

(a) By the definition of covariant derivative on a surface and the hypotheses about α, we have

$$\begin{aligned}\nabla_{E_1} V &= \nabla_{\alpha'} V \\ &= (V(\alpha))' \\ &= V'.\end{aligned}$$

The last step is simply what is meant by V'.

(b) By definition, α is a geodesic if, and only if, its acceleration is normal to M. We must, therefore, calculate $\alpha'' = E_1'$.

Applying the above result, we have

$$E_1' = \nabla_{E_1} E_1.$$

Now we can apply the connection equations to get

$$\begin{aligned}E_1' &= \nabla_{E_1} E_1 \\ &= \omega_{12}(E_1)\, E_2 + \omega_{13}(E_1)\, E_3.\end{aligned}$$

Since E_3 is normal to M, α is geodesic precisely when the E_2 term in the acceleration is zero, that is, when

$$\omega_{12}(E_1) = 0.$$

Solution 1.2

The required normal curvature is

$$S(E_1) \cdot E_1 = -\nabla_{E_1} E_3 \cdot E_1.$$

From the connection equations,

$$\begin{aligned}\nabla_{E_1} E_3 &= \omega_{31}(E_1) E_1 + \omega_{32}(E_1) E_2 \\ &= -\omega_{13}(E_1) E_1 - \omega_{23}(E_1) E_2.\end{aligned}$$

Hence

$$\begin{aligned}S(E_1) \cdot E_1 &= -(-(\omega_{13}(E_1) E_1 - \omega_{23}(E_1) E_2)) \cdot E_1 \\ &= \omega_{13}(E_1),\end{aligned}$$

as required.

Note: You could have shortened the working slightly by applying Corollary 1.5 directly.

Solution 1.3

(a) First, we calculate the partial velocities.

$$\mathbf{x}_u = (-r \sin u \cos v, -r \sin u \sin v, r \cos u),$$

$$\mathbf{x}_v = (-(R + r\cos u) \sin v, (R + r \cos u) \cos v, 0).$$

We also have

$$\|\mathbf{x}_u\| = r,\ \|\mathbf{x}_v\| = (R + r \cos u).$$

Thus

$$E_1 = (-\sin u \cos v, -\sin u \sin v, \cos u),$$

$$E_2 = (-\sin v, \cos v, 0)$$

are certainly unit length. It is also clear that

$$E_1 \cdot E_2 = 0,$$

so they are orthogonal. The third member can now be defined by

$$\begin{aligned}E_3 &= E_1 \times E_2 \\ &= (-\cos u \cos v, -\cos u \sin v, -\sin u).\end{aligned}$$

(b) Since du and dv pick out the $\mathbf{x}_u$ and $\mathbf{x}_v$ components respectively of a vector tangent to T, we have

$$\begin{aligned}du(E_1) &= du\left(\frac{\mathbf{x}_u}{\|\mathbf{x}_u\|}\right) \\ &= \frac{1}{r},\\ du(E_2) &= du\left(\frac{1}{R + r\cos u}\mathbf{x}_v\right) \\ &= 0,\\ dv(E_1) &= dv\left(\frac{1}{r}\mathbf{x}_u\right) \\ &= 0,\\ dv(E_2) &= dv\left(\frac{1}{R + r\cos u}\mathbf{x}_v\right) \\ &= \frac{1}{R + r\cos u}.\end{aligned}$$

Comparing these with

$$\theta_i(E_j) = \delta_{ij},$$

we can see that

$$du = \frac{1}{r}\theta_1,$$

$$dv = \frac{1}{R + r\cos u}\theta_2.$$

The required results follow immediately.

(c) Differentiating

$$\begin{aligned}d\theta_1 &= dr\, du \\ &= 0 du \\ &= 0,\\ d\theta_2 &= d(R + r\cos u)\, dv \\ &= -r \sin u\, du\, dv \\ &= -r\sin u \frac{\theta_1}{r} \wedge \frac{\theta_2}{R + r\cos u} \\ &= -\frac{\sin u}{R + r\cos u}\theta_1 \wedge \theta_2.\end{aligned}$$

Substituting into

$$d\theta_1 = \omega_{12} \wedge \theta_2,$$

we have

$$\begin{aligned}0 &= (a\theta_1 + b\theta_2) \wedge \theta_2 \\ &= a\theta_1 \wedge \theta_2 + 0 \\ &= a\theta_1 \wedge \theta_2.\end{aligned}$$

Hence $a = 0$ and $\omega_{12} = b\theta_2$.

Using the second equation

$$\begin{aligned}d\theta_2 &= -\frac{\sin u}{R + r\cos u}\theta_1 \wedge \theta_2 \\ &= -b\theta_2 \wedge \theta_1 \\ &= b\theta_1 \wedge \theta_2.\end{aligned}$$

Hence

$$b = -\frac{\sin u}{R + r\cos u}$$

and

$$\omega_{12} = -\frac{\sin u}{R + r\cos u}\theta_2.$$

(d) Using the hint, we have

$$\begin{aligned} d\omega_{12} &= d\left(-\frac{\sin u}{R+r\cos u}\right)\theta_2 + \left(-\frac{\sin u}{R+r\cos u}\right)d\theta_2 \\ &= -\frac{\cos u(R+r\cos u)-\sin u(-r\sin u)}{(R+r\cos u)^2}du\,\theta_1 \\ &\quad + \left(-\frac{\sin u}{R+r\cos u}\right)\left(-\frac{\sin u}{R+r\cos u}\right)\theta_1\wedge\theta_2 \\ &= -\frac{\cos u}{r(R+r\cos u)}\theta_1\wedge\theta_2. \end{aligned}$$

Here we have made use of the result for $d\theta_2$ and also

$$\theta_1 = r\,du.$$

Solution 2.1

In the previous section we found that

$$d\omega_{12} = -\frac{\cos u}{r(R+r\cos u)}\theta_1\wedge\theta_2.$$

Comparing this with

$$d\omega_{12} = -K\theta_1\wedge\theta_2,$$

we see that

$$K = \frac{\cos u}{r(R+r\cos u)}.$$

Solution 2.2

Yes.

You may well recall that *O'Neill* has proved that the meridians and parallels of any surface of revolution are principal curves. However, since the unit normal vector field E_3 that we obtained is fairly simple, direct differentiation shows that E_1 and E_2 are principal as follows.

$$\begin{aligned} S(E_1) &= -\nabla_{E_1}E_3 \\ &= -\frac{1}{\|\mathbf{x}_u\|}\nabla_{\mathbf{x}_u}E_3 \quad \text{(definition of } E_1 \text{ and linearity)} \\ &= -\frac{1}{\|\mathbf{x}_u\|}\frac{\partial E_3}{\partial u} \\ &= -\frac{1}{\|\mathbf{x}_u\|}(\sin u\cos v, \sin u\sin v, -\cos u) \\ &= \frac{1}{\|\mathbf{x}_u\|}E_1 = \frac{1}{r}E_1. \end{aligned}$$

Thus E_1 is an eigenvector of the shape operator and so is principal.

The calculations for $S(E_2)$ are similar.

Solution 3.1

We have

$$\mathbf{x}_u = (-r\sin u\cos v, -r\sin u\sin v, r\cos u),$$
$$\mathbf{x}_v = (-(R+r\cos u)\sin v, (R+r\cos u)\cos v, 0),$$

and so

$$\begin{aligned} E &= r^2, \\ \sqrt{E} &= r, \\ F &= 0, \quad \text{(orthogonal patch)} \\ G &= (R+r\cos u)^2 \\ \sqrt{G} &= (R+r\cos u) \\ \frac{1}{\sqrt{EG}} &= \frac{1}{r(R+r\cos u)}. \end{aligned}$$

Hence

$$\begin{aligned} (\sqrt{G})_u &= -r\sin u, \\ (\sqrt{E})_v &= 0, \\ \frac{(\sqrt{G})_u}{\sqrt{E}} &= -\sin u, \\ \left(\frac{(\sqrt{G})_u}{\sqrt{E}}\right)_u &= -\cos u, \\ \left(\frac{(\sqrt{E})_v}{\sqrt{G}}\right)_v &= 0. \end{aligned}$$

Substitution now gives

$$K = \frac{\cos u}{r(R+r\cos u)}.$$

Solution 3.2

(a) Using the calculations above, we have

$$\begin{aligned} \theta_1 &= \sqrt{E}\,du \\ &= r\,du. \\ \theta_2 &= \sqrt{G}\,dv \\ &= (R+r\cos u)dv. \end{aligned}$$

(b) We have

$$\begin{aligned} d\theta_1 &= dr\,du \\ &= 0, \\ d\theta_2 &= d(R+r\cos u)dv \\ &= (-r\sin u\,du + 0\,dv)dv \\ &= -r\sin u\,du\,dv. \end{aligned}$$

(c) Substituting in

$$d\theta_1 = \omega_{12}\wedge\theta_2$$

gives

$$\begin{aligned} 0 &= (a\,du + b\,dv)\wedge((R+r\cos u)\,dv) \\ &= a(R+r\cos u)\,du\,dv. \end{aligned}$$

It follows that $a = 0$.

Substituting into

$$d\theta_2 = -\omega_{12}\wedge\theta_1$$

gives

$$\begin{aligned} -r\sin u\,du\,dv &= -(a\,du + b\,dv)\wedge(r\,du) \\ &= 0 - rb\,dv\,du \\ &= rb\,du\,dv. \end{aligned}$$

Thus $b = -\sin u$ and

$$\omega_{12} = -\sin u\,dv.$$

(d) From the last answer

$$\begin{aligned} d\omega_{12} &= d(-\sin u)\,dv \\ &= -\cos u\,du\,dv. \end{aligned}$$

(e) We substitute into the second structural equation.

$$\begin{aligned} d\omega_{12} &= -\cos u\,du\,dv \\ &= -K\theta_1\wedge\theta_2 \\ &= -K(r\,du)\wedge((R+r\cos u)\,dv) \\ &= -Kr(R+r\cos u)\,du\,dv. \end{aligned}$$

Hence

$$-Kr(R+r\cos u) = -\cos u$$

and so

$$K = \frac{\cos u}{r(R+r\cos u)}.$$

Solution 4.1

Since we shall require them, we calculate all the partial velocities to begin with. We suppress the parameters.

$$\mathbf{x}_u = (-r \sin u \cos v, -r \sin u \sin v, r \cos u),$$
$$\mathbf{x}_v = (-(R + r \cos u) \sin v, (R + r \cos u) \cos v, 0),$$
$$\mathbf{y}_u = (-\sin u \cos v, -\sin u \sin v, \cos u),$$
$$\mathbf{y}_v = (-\cos u \sin v, \cos u \cos v, 0).$$

(a) We have

$$\mathbf{x}_u \times \mathbf{x}_v = r(R + r \cos u) \times (-\cos u \cos v, -\cos u \sin v, -\sin u).$$

We are specifically asked for the outward normal. By looking at the special case $u = v = 0$, we obtain the possibilities

$$\pm(1, 0, 0)$$

for the normal at the point

$$(R + r, 0, 0)$$

which is on the outer 'equator' of the torus. Thus choosing the positive sign gives the outward normal which will be

$$U = (\cos u \cos v, \cos u \sin v, \sin u).$$

(b) Since

$$U(\mathbf{x}(u, v)) = \mathbf{y}(u, v),$$

F does define the Gauss map.

(c) From the calculations above, we can see that

$$\mathbf{x}_u \times \mathbf{x}_v = -r(R + r \cos u)U,$$

so

$$\begin{aligned}\eta(\mathbf{x}_u, \mathbf{x}_v) &= \mathbf{x}_u \times \mathbf{x}_v \cdot U \\ &= -r(R + r \cos u)U \cdot U \\ &= -r(R + r \cos u).\end{aligned}$$

From the expressions for the partial velocities of $\mathbf{y}$, and the fact that U and V have the same coordinate expressions, we have

$$\begin{aligned}\mathbf{y}_u \times \mathbf{y}_v &= (-\cos^2 u \cos v, -\cos^2 u \sin v, -\cos u \sin u) \\ &= -\cos u\, V.\end{aligned}$$

Hence

$$\begin{aligned}\zeta(\mathbf{y}_u, \mathbf{y}_v) &= \mathbf{y}_u \times \mathbf{y}_v \cdot V \\ &= -\cos u\, V \cdot V \\ &= -\cos u.\end{aligned}$$

(d) Using the definitions

$$\begin{aligned}F^*\zeta(\mathbf{x}_u, \mathbf{x}_v) &= \zeta(F_*(\mathbf{x}_u), F_*(\mathbf{x}_v)) \\ &= \zeta(\mathbf{y}_u, \mathbf{y}_v) \\ &= -\cos u.\end{aligned}$$

From what we have calculated so far, we can say that

$$F^*\zeta(\mathbf{x}_u, \mathbf{x}_v) = \frac{\cos u}{r(R + r \cos u)} \eta(\mathbf{x}_u, \mathbf{x}_v).$$

It follows that

$$F^*\zeta = \frac{\cos u}{r(R + r \cos u)} \eta$$

and so

$$K = \frac{\cos u}{r(R + r \cos u)}.$$

Note: If you do not choose the outward normal, but do everything else consistently, you would arrive at the same result.

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